MODULAR ELEMENTS
OF THE LATTICE OF SEMIGROUP VARIETIES

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Abstract. We characterize semigroup varieties being both modular and lower-modular elements of the lattice of all semigroup varieties. As a corollary, we classify all neutral elements of the lattice.

INTRODUCTION AND SUMMARY

Unlike lattices of group or ring varieties, the lattice \textbf{SEM} of all semigroup varieties does not fulfil the modular law. (This fact was discovered in the late 1960s by Schwabauer [19,20] and Ježek [6].) Thus, if the lattice formula

\[(x, y, z) \iff (x \leq z \rightarrow (x \lor y) \land z = x \lor (y \land z))\]

holds true when the variables \(x, y, z\) are evaluated at some \(X, Y, Z \in \text{SEM}\), then the varieties \(X, Y, Z\) should enjoy some specific structural and/or equational properties. Studying such ‘modularity-related’ properties of semigroup varieties constitutes a natural research programme, and a considerable amount of work has been done in this direction. For instance, the semigroup varieties whose subvarieties satisfy (1) have been classified by the author [24–27] (see also the recent papers [21, 23, 28] by Vernikov and the author for a modernized proof and generalizations). The formula (1) is then treated as a quasi-identity, that is, the three variables are thought to be bound by universal quantifiers. Yet another possible approach within the same framework consists of considering (1) as an open formula when only two variables are subject to universal quantification while one variable is left free. It is the approach that we adopt in the present note.

The first, the second, and the third variables of the formula (1) play different roles, and therefore, quantifying two of them we get three in general different
notions. In the literature it is common to call an element \( y \) of a lattice \( \langle L, \lor, \land \rangle \) modular if the formula \( M(x, y, z) \) holds true for every choice of \( x, z \in L \). In other words, \( y \in L \) is a modular element if it makes the formula
\[
\mathcal{C}M(y) \equiv \left( \forall x, z \ M(x, y, z) \right)
\]
hold true. Similarly, we define the formulas
\[
\mathcal{U}M(z) \equiv \left( \forall x, y \ M(x, y, z) \right),
\]
\[
\mathcal{L}M(x) \equiv \left( \forall y, z \ M(x, y, z) \right).
\]
We call an element \( x \in L \) such that \( \mathcal{L}M(x) = \text{true} \) lower-modular, and an element \( z \in L \) such that \( \mathcal{U}M(z) = \text{true} \) is called upper-modular.

Modular elements of the lattice \( \text{SEM} \) have been studied by Ježek and McKenzie [8] even though a complete classification of such elements has not been the main goal of [8] (and has not been achieved). In the present note we describe semigroup varieties being both modular and lower-modular elements of \( \text{SEM} \). This result has an interesting application to the problem of first order definability within \( \text{SEM} \). Namely, it turns out that an important class of semigroup varieties whose definability has played a crucial role in Ježek and McKenzie’s studies (and has been established in [8] by means of a complicated and rather artificial formula) is in fact defined in \( \text{SEM} \) in a natural and fairly transparent way. Amongst other applications, we mention a classification of the neutral elements of the lattice \( \text{SEM} \).

The note is organized as follows. In order to make it to a reasonable extent self-contained, we provide in Sections 1 and 2 all necessary preliminaries from respectively lattice theory and the theory of semigroup varieties. Section 3 is devoted to a description of varieties that are both modular and lower-modular elements of the lattice \( \text{SEM} \). A few corollaries and applications of this description are collected in Section 4.

### 1. Preliminaries on lattices

In the introduction we defined three notions of modularity of elements in a lattice. First of all, we discuss these notions in more detail.

Let \( \langle L, \lor, \land \rangle \) be a lattice and \( a, b, c, d, e \in L \). We write \( \text{Pent}(a, b, c, d, e) \) if the elements \( a, b, c, d, e \) form a pentagon (that is, a five-element non-modular sublattice of \( L \)) in which \( c \) is the least element, \( d \) is the greatest element, \( a < b \) and \( e \lor a = d, \ e \land b = c \) (see Fig. 1). The following observation (cf. [7, Proposition 2.1]) gives a ‘pictorial’ characterization of modular elements:

**Proposition 1.1.** An element \( y \) of a lattice \( L \) is modular if and only if there exist no elements \( a, b, c, d \in L \) such that \( \text{Pent}(a, b, c, d, y) \). \( \square \)
We also observe that the notion of a modular element is self-dual in the sense that a modular element of lattice $L$ is also modular in the dual of $L$. (This readily follows from the definition or from Proposition 1.1.) Now we may confess that the paper [8] by Ježek and McKenzie has dealt with the lattice of equational theories of semigroups, that is, the dual of $\text{SEM}$ rather than the lattice $\text{SEM}$ itself. However, the modular elements of the former lattice precisely correspond to the modular elements of $\text{SEM}$.

Clearly, the notion of a lower-modular element and that of an upper-modular elements are dual to each other. As for a ‘pictorial’ characterization in the flavour of Proposition 1.1, the things are a little bit more complicated. It is true that lower-modular and upper-modular elements cannot be included into a pentagon at certain positions: for instance, if $x$ is a lower-modular element of a lattice $L$ then there exist no elements $b, c, d, e \in L$ such that $\text{Pent}(x, b, c, d, e)$. However the converse, generally speaking, is not true. Indeed, in the 7-element lattice shown on Fig. 2 the element $g$ is not lower-modular but at the same time there are no elements $x, y, z, t$ such that $\text{Pent}(g, x, y, z, t)$. Fortunately, this turns out to be the only exception.

We call the lattice on Fig. 2 the lower heptagon. If $\langle L, \lor, \land \rangle$ is a lattice and $a, b, c, d, e, f, g \in L$, we write $\text{LowHept}(a, b, c, d, e, f, g)$ if the elements $a, b, c, d, e, f, g$ form a lower heptagon in which $\text{Pent}(a, b, c, d, e)$ and, besides that, $f$ is the least element, $g < a$, $e \land g = f$, and $g \lor e = a$. 
Proposition 1.2. An element \( x \) of a lattice \( L \) is lower-modular if and only if there exist no elements \( a, b, c, d, e, g \in L \) such that either \( \text{Pent}(x, b, c, d, e) \) or \( \text{LowHept}(a, b, c, d, e, f, x) \).

Proof. This proposition (as well as the previous one) easily follows from a well known description of the free product of the 1-element and the 2-element lattices (see, e.g., [5, Section I.2]). However, for the sake of completeness, we do provide a short self-contained proof of sufficiency (necessity is clear).

Thus, let \( x \in L \) fail to be lower-modular and let \( y, z \in L \) 'witness' this fact, that means \( x \leq z \) and \( (x \lor y) \land z \neq x \lor (y \land z) \). Then clearly
\[
x \leq x \lor (y \land z) < (x \lor y) \land z,
\]
and it is easy to see that \( \text{Pent}(x \lor (y \land z), (x \lor y) \land z, y \land z, x \lor y, y) \). Thus, if \( x = x \lor (y \land z) \) we have \( \text{Pent}(x, b, c, d, e) \) with \( b = (x \lor y) \land z, c = y \land z, d = x \lor y, \) and \( e = y \). On the other hand, if \( x < x \lor (y \land z) \), then we have \( \text{LowHept}(a, b, c, d, e, f, x) \) with \( b, c, d, e \) as above, \( a = x \lor (y \land z), f = x \lor y \). \( \Box \)

Our next observation is rather straightforward:

Lemma 1.3. If \( x_1 \) and \( x_2 \) are lower-modular elements of a lattice \( L \), then the join \( x_1 \lor x_2 \) also is lower-modular.

Proof. Let \( y, z \in L \) be such that \( x_1 \lor x_2 \leq z \). Then \( x_1 \leq z \) and \( x_2 \leq z \), and we have
\[
(x_1 \lor x_2) \lor (y \land z) = x_1 \lor (x_2 \lor (y \land z))
\]
\[
= x_1 \lor ((x_2 \lor y) \land z) \text{ because } x_2 \text{ is lower-modular}
\]
\[
= (x_1 \lor (x_2 \lor y)) \land z \text{ because } x_1 \text{ is lower-modular}
\]
\[
= ((x_1 \lor x_2) \lor y) \land z.
\]
\( \Box \)

For any non-empty set \( A \) we denote by \( \text{Part}(A) \) the lattice of all partitions of \( A \) (cf. [5, Section IV.4]). We shall make use of the following description of modular and upper-modular elements of this lattice:

Proposition 1.4. For a partition \( \pi \) of a set \( A \) the following are equivalent:

(i) \( \pi \) is a modular element of the lattice \( \text{Part}(A) \);
(ii) \( \pi \) is an upper-modular element of the lattice \( \text{Part}(A) \);
(iii) \( \pi \) has at most one non-singleton block.

Proposition 1.4 easily follows from results of Ore’s classical paper [14] of 1942. (In that paper Ore completely described all pairs \( \sigma, \tau \) of partitions such that the formula \( M(\rho, \sigma, \tau) \) holds true for every \( \rho \in \text{Part}(A) \).) Much later the equivalences (i) \( \Leftrightarrow \) (iii) and (ii) \( \Leftrightarrow \) (iii) were separately rediscovered and reproved in respectively [7, Proposition 2.2] and [22, Proposition 3].
Recall that an element $x$ of a lattice $(L; \lor, \land)$ is called *distributive* if
$$\forall y, z \quad x \lor (y \land z) = (x \lor y) \land (x \lor z),$$
codistributive if
$$\forall y, z \quad x \land (y \lor z) = (x \land y) \lor (x \land z),$$
and *neutral* if
$$\forall y, z \quad (x \lor y) \land (y \lor z) \land (z \lor x) = (x \lor y) \lor (y \lor z) \lor (z \lor x).$$
Clearly, each distributive [codistributive] element is lower-modular [upper-modular]. The next proposition collects two well known characterizations of neutral elements [5, Theorem III.2.4], see also [9, Lemma 1.3].

**Proposition 1.5.** For an element $x$ of a lattice $L$ the following are equivalent:

(i) $x$ is neutral;

(ii) $x$ is distributive, codistributive and modular;

(iii) for all $y, z \in L$ the sublattice generated by $x, y, z$ is distributive. $\square$

We will also need two further observations in the flavour of Lemma 1.3:

**Lemma 1.6.** Let $a$ be a neutral element of a lattice $L$.

(i) If $x \in L$ is neutral, then so is $x \lor a$.

(ii) If $y \in L$ is modular, then so is $y \lor a$.

*Proof.* Part (i) is well known (cf., e.g., [5, Theorem III.2.9]). In order to verify (ii), let $x, z \in L$ be such that $x \leq z$. Then
$$x \lor ((y \lor a) \land z) = x \lor ((y \land z) \lor (a \land z)) \quad \text{because } a \text{ is neutral}$$
$$= (x \lor (y \land z)) \lor (a \land z) \quad \text{because } y \text{ is modular}$$
$$= ((x \lor y) \land z) \lor (a \land z) \quad \text{because } a \text{ is neutral}$$
$$= ((x \lor y) \lor a) \land z \quad \text{because } a \text{ is neutral}$$
$$= (x \lor (y \lor a)) \land z. \quad \square$$

2. Preliminaries on semigroup varieties

We assume the reader’s acquaintance with some rudiments of universal algebra such as the HSP-theorem (cf., e.g., [2, Chapter 2]). On the other hand, as far as semigroup notions are concerned, we have tried to keep the presentation on a fairly elementary level with almost no background presumed. The classic survey by Evans [4] may be useful for those readers who would want to place the results of the present note into a broader context.

Let $S$ be a semigroup. A non-empty subset $I \subseteq S$ is said to be an *ideal* of $S$ if $si, is \in I$ for all $i \in I$, $s \in S$. The *Rees congruence* $\rho_I$ induced by $I$ is the partition of $S$ for which $I$ is one of the blocks and all other blocks are
singleton. One usually writes $S/I$ for the quotient semigroup $S/\rho_I$. Observe that Proposition 1.4 implies that each Rees congruence is a modular and an upper-modular element of the lattice of all congruences on $S$ as the latter constitutes a sublattice in the lattice $\text{Part}(S)$.

The collection $\text{SEM}$ of all semigroup varieties constitutes a lattice under inclusion. The lattice operations in $\text{SEM}$ will be denoted by $\lor$ and $\land$: recall that $\mathcal{U} \land \mathcal{W}$ is nothing but the intersection of the varieties $\mathcal{U}$ and $\mathcal{W}$ while $\mathcal{U} \lor \mathcal{W}$ is the least variety containing both $\mathcal{U}$ and $\mathcal{W}$, that is,

$$\mathcal{U} \lor \mathcal{W} = \text{HSP} (\mathcal{U} \cup \mathcal{W}).$$

Given $\mathcal{U} \in \text{SEM}$, we denote by $L(\mathcal{U})$ the lattice of all subvarieties of $\mathcal{U}$.

The lattice $\text{SEM}$ is dual to the lattice of all fully invariant congruences on the absolutely free semigroup $F$ over a countably infinite set of generators (the latter lattice is also known as the lattice of equational theories of semigroups). Suppose that $I$ is a fully invariant ideal of $F$ (this means that $I \varphi \subseteq I$ for every endomorphism $\varphi$ of $F$). Then it is easy to see that the Rees congruence $\rho_I$ is fully invariant as well. We call a variety $\mathcal{R} \in \text{SEM}$ a Rees variety if the fully invariant congruence corresponding to $\mathcal{R}$ is a Rees congruence. (In the literature such varieties were sometimes referred to as $0$-reduced, see, e.g., [10].) The aforementioned property of Rees congruences leads to the following

**Proposition 2.1.** [22, Corollary 3] Each Rees variety is a modular and a lower-modular element of $\text{SEM}$. □

By $\text{var} \Sigma$ we denote the variety of all semigroups satisfying the identity system $\Sigma$. Let $w \in F$ be a term. We adopt the usual agreement of writing $w = 0$ for the identity system $\{wu = uw = w \mid u \in F\}$ and referring to such an expression as to a single identity. In the sequel we make use of the following straightforward characterization of Rees varieties:

**Proposition 2.2.** A variety $\mathcal{R} \in \text{SEM}$ is Rees if and only if $\mathcal{R} = \text{var} \Sigma$ where $\Sigma$ consists of identities of the form $w = 0$. □

Recall that a semigroup $S$ with zero 0 is said to be a nilsemigroup if for every $s \in S$ there exists a positive integer $n$ such that $s^n = 0$. A semigroup variety $\mathcal{R}$ consisting of nilsemigroups is called a nil-variety. It is easy to see that $\mathcal{R} \in \text{SEM}$ is a nil-variety if and only if $\mathcal{R}$ is contained in a Rees variety.

If $u \in F$ is a term, then $c(u)$ denotes the set of all free generators occurring in $u$. The following technical remark is obvious and well known:

**Lemma 2.3.** If a nil-variety $\mathcal{R}$ satisfies an identity $u = v$ with $c(u) \neq c(v)$, then $\mathcal{R}$ satisfies also the identity $u = 0$. □
Every non-trivial semigroup variety contains an atom, that is, a minimal non-trivial subvariety. The atoms of $\text{SEM}$ are well known (cf. [4, Section IV]). Two of them that play a distinguished role in the present note are the variety $\mathfrak{SL} = \text{var}\{x^2 = x, \ xy = yx\} = \text{var}\{u = v \mid c(u) = c(v)\}$
of all semilattices (commutative idempotent semigroups) and the variety $\mathfrak{ZM} = \text{var}\{xy = 0\} = \text{var}\{u = v \mid \ell(u), \ell(v) > 1\}$
of all zero semigroups. Their most important property is contained in the following

**Proposition 2.4.** The atoms $\mathfrak{SL}$ and $\mathfrak{ZM}$ are neutral elements of the lattice $\text{SEM}$ of all semigroup varieties.

**Proof.** Rather than proceeding with a direct proof (which is not difficult, by the way), we show how these facts can be deduced from certain results by Mel’nik. It is quite common in the literature to merely refer to Mel’nik’s papers [12] (for $\mathfrak{SL}$) and [13] (for $\mathfrak{ZM}$) but the papers do not explicitly contain the claims of Proposition 2.4, and moreover, do not suffice to justify these claims. In fact, in [12] it only has been shown that the mapping $\xi_{\mathfrak{SL}} : \mathfrak{U} \mapsto \mathfrak{SL} \lor \mathfrak{U}$
is an endomorphism of the lattice $\text{SEM}$. Clearly, this means that $\mathfrak{SL}$ is a distributive element of $\text{SEM}$. Earlier Mel’nik [11] proved that the restriction of $\xi_{\mathfrak{SL}}$ to every lattice $L(\mathfrak{W})$ with $\mathfrak{W} \not\supseteq \mathfrak{SL}$ is injective (the same result was independently discovered by Salii [18]). It is easy to see that the latter property is equivalent to the fact that $\mathfrak{SL}$ is a modular element of $\text{SEM}$. In view of Proposition 1.5, the only ingredient which one needs to conclude that the atom $\mathfrak{SL}$ is neutral is its codistributivity in $\text{SEM}$. However, nor this ingredient neither any of its equivalents were mentioned in [11, 12]. It seems that a statement equivalent to the fact that each atom of $\text{SEM}$ is a codistributive element first appeared in [1].

Similarly, in [13] Mel’nik proved that the mapping $\xi_{\mathfrak{ZM}} : \mathfrak{U} \mapsto \mathfrak{ZM} \lor \mathfrak{U}$
is an endomorphism of $\text{SEM}$ and that the restriction of $\xi_{\mathfrak{ZM}}$ to every lattice $L(\mathfrak{W})$ with $\mathfrak{W} \not\supseteq \mathfrak{ZM}$ is injective. As discussed above, this yields that $\mathfrak{ZM}$ is both a distributive and a modular element of $\text{SEM}$. The codistributivity of $\mathfrak{ZM}$ follows from the aforementioned result of [1].

Combining Propositions 2.1 and 2.4 with Lemmas 1.3 and 1.6, we obtain

**Corollary 2.5.** For each Rees variety $\mathfrak{R}$, the join $\mathfrak{R} \lor \mathfrak{SL}$ is a modular and a lower-modular element of the lattice $\text{SEM}$. □
The claim of Corollary 2.5 concerning modular elements was first established in [8, Proposition 1.1]. As we have demonstrated, it is a straightforward consequence of very basic principles. In contrast, the partial conversion of this claim obtained in [8, Proposition 1.6] is rather non-trivial. In its formulation, a proper variety means a variety not equal to the class of all semigroups.

**Proposition 2.6.** If a proper variety \( \mathcal{V} \) is a modular element of the lattice \( \text{SEM} \), then \( \mathcal{V} \subseteq \mathcal{R} \lor \mathcal{S}\mathcal{L} \) for some Rees variety \( \mathcal{R} \). \( \square \)

3. Varieties that are both modular and lower-modular

**Theorem 3.1.** A variety \( \mathcal{V} \) is a modular and lower-modular element of the lattice \( \text{SEM} \) if and only if \( \mathcal{V} \) satisfies one of the following conditions:

(i) \( \mathcal{V} \) coincides with the class of all semigroups;
(ii) \( \mathcal{V} \) is a Rees variety;
(iii) \( \mathcal{V} = \mathcal{R} \lor \mathcal{S}\mathcal{L} \) for some Rees variety \( \mathcal{R} \).

**Proof. Necessity.** Of course, we may assume that \( \mathcal{V} \) is a proper variety. By Proposition 2.6 we then have \( \mathcal{V} \subseteq \mathcal{R}' \lor \mathcal{S}\mathcal{L} \) for some Rees variety \( \mathcal{R}' \). Using this inclusion and the fact that the variety \( \mathcal{S}\mathcal{L} \) is a neutral element of \( \text{SEM} \), we can decompose \( \mathcal{V} \) as follows:

\[
(2) \quad \mathcal{V} = \mathcal{V} \land (\mathcal{R}' \lor \mathcal{S}\mathcal{L}) = (\mathcal{V} \land \mathcal{R}') \lor (\mathcal{V} \land \mathcal{S}\mathcal{L}).
\]

Let \( \mathcal{R} = \mathcal{V} \land \mathcal{R}' \); being a subvariety of a Rees variety \( \mathcal{R} \) is a nil-variety. Since \( \mathcal{S}\mathcal{L} \) is an atom, (2) means that either \( \mathcal{V} = \mathcal{R} \) or \( \mathcal{V} = \mathcal{R} \lor \mathcal{S}\mathcal{L} \). Therefore it remains to verify that \( \mathcal{R} \) is a Rees variety. We denote by \( \mathcal{R} \) the least Rees variety containing \( \mathcal{R} \). From Proposition 2.2 it follows that the variety \( \mathcal{R} \) can be defined by the set of all identities of the form \( w = 0 \) that hold true in \( \mathcal{R} \).

Now, arguing by contradiction, suppose that \( \mathcal{R} \) is not a Rees variety, in other words, \( \mathcal{R} \subsetneq \mathcal{R} \). Then there exists an identity \( u = v \) that holds in \( \mathcal{R} \) but fails in \( \mathcal{R} \); let us fix such an identity. Observe that then the identity \( u = 0 \) cannot hold in \( \mathcal{R} \); this by Lemma 2.3 implies \( c(u) = c(v) \). Therefore the identity \( u = v \) holds in the variety \( \mathcal{S}\mathcal{L} \) whence we can conclude that it also holds in \( \mathcal{V} = \mathcal{R} \lor (\mathcal{V} \land \mathcal{S}\mathcal{L}) \).

The absolutely free semigroup \( F \) is known to be a subdirect product of finite groups whence every non-trivial semigroup identity must fail in a suitable finite group. Let \( G \) be a finite group that does not satisfy the chosen identity \( u = v \) and let \( \mathfrak{S} \) be the semigroup variety generated by \( G \). Finally, consider the variety \( \mathfrak{W} = \mathcal{R} \lor (\mathcal{V} \land \mathcal{S}\mathcal{L}) \). Clearly, \( \mathcal{V} \subseteq \mathfrak{W} \). We aim to prove that

\[
(3) \quad \mathcal{V} \lor (\mathfrak{S} \land \mathfrak{W}) \neq (\mathcal{V} \lor \mathfrak{S}) \land \mathfrak{W}
\]

thus showing that \( \mathfrak{W} \) fails to be lower-modular and arriving at the desired contradiction.
Observe that the left-hand side of (3) is equal to $\mathfrak{W}$ because the meet $\mathfrak{G} \wedge \mathfrak{W}$ is easily seen to be trivial. Therefore the identity $u = v$ holds in the left-hand side of (3), and it remains to construct a semigroup that belongs to the right-hand side of (3) and does not satisfy $u = v$. We start our construction with a nilsemigroup $N \in \mathfrak{N}$ such that $N$ does not satisfy the identity $u = 0$. In the direct product $N \times G$ we consider the set $I = \{(0, g) \mid g \in G\}$. Obviously, $I$ is an ideal of $N \times G$. Let $S = (N \times G) / I$. Then $S$ belongs to the join $\mathfrak{W} \vee \mathfrak{G} = \text{HSP}(\mathfrak{U} \cup \mathfrak{F})$ as a homomorphic image of a direct product of members of $\mathfrak{W}$ and $\mathfrak{G}$. Next we aim to show that $S$ belongs to $\mathfrak{R}$ and thus to $\mathfrak{W}$. By the above characterization of $\mathfrak{R}$, it suffices to show that $S$ satisfies every identity of the form $w = 0$ that holds in $\mathfrak{N}$. Indeed, for any $k$-ary term $w \in F$, evaluating it at some elements $(n_1, g_1), \ldots, (n_k, g_k) \in N \times G$ returns the pair $(w(n_1, \ldots, n_k), w(g_1, \ldots, g_k))$. If the term is such that the identity $w = 0$ holds in $\mathfrak{N}$, then

$$w((n_1, g_1), \ldots, (n_k, g_k)) = (0, w(g_1, \ldots, g_k)) \in I$$

for every $k$-tuple $((n_1, g_1), \ldots, (n_k, g_k))$ of pairs from $N \times G$. Therefore the image of $w((n_1, g_1), \ldots, (n_k, g_k))$ in $S = (N \times G) / I$ is equal to 0, in other words, the identity $w = 0$ holds true in $S$. Thus, $S$ belongs to $\mathfrak{R}$ whence $S$ belongs to the right-hand side of (3).

It remains to verify that the semigroup $S$ does not satisfy the identity $u = v$. If $k$ is the arity of the term $u$, then the term $v$ is $k$-ary too because of $c(u) = c(v)$. By our choice of the nilsemigroup $N$ and the group $G$ there exist $n_1, \ldots, n_k \in N$ such that

$$u(n_1, \ldots, n_k) \neq 0,$$

and $g_1, \ldots, g_k \in G$ such that

$$u(g_1, \ldots, g_k) \neq v(g_1, \ldots, g_k).$$

Now calculating in the direct product $N \times G$, we see that

$$u((n_1, g_1), \ldots, (n_k, g_k)) \notin I$$

because of (4) and

$$u((n_1, g_1), \ldots, (n_k, g_k)) \neq v((n_1, g_1), \ldots, (n_k, g_k))$$

because of (5). Since all blocks of the Rees congruence $\rho_I$ except $I$ are singletons, we conclude that the image of $u((n_1, g_1), \ldots, (n_k, g_k))$ in $S$ is different from the image of $v((n_1, g_1), \ldots, (n_k, g_k))$. This means that the images of the pairs $(n_1, g_1), \ldots, (n_k, g_k)$ in $S$ violate the identity $u = v$. Thus, necessity is established.

Sufficiency follows from Proposition 2.1 and Corollary 2.5. \qed
Corollaries and applications

4.1. Strongly modular and neutral elements of SEM. Theorem 3.1 and its proof admit a number of corollaries that shed some new light on the structure of the lattice SEM as a whole. In order to formulate the first of these corollaries we need a notion that, though natural enough, appears to have been overlooked so far. Namely, we call an element $x$ of a lattice $L$ strongly modular if $x$ is modular, lower-modular and upper-modular in $L$. From the description of the free product of the 1-element and the 2-element lattices mentioned in Section 1, one can easily deduce the following characterization of such elements: $x \in L$ is strongly modular if and only if for all $y, z \in L$ such that two of the elements $x, y, z$ are comparable the sublattice of $L$ generated by $x, y, z$ is modular. It is also clear that in a modular lattice every element is strongly modular. We will see that the lattice SEM is ‘strongly non-modular’ in the sense that it contains almost no strongly modular elements.

**Proposition 4.1.** For a variety $\mathcal{V} \in \text{SEM}$ the following are equivalent:

(i) $\mathcal{V}$ is a neutral element of SEM;
(ii) $\mathcal{V}$ is a strongly modular element of SEM;
(iii) $\mathcal{V}$ either is trivial or is equal to the class of all semigroups or coincides with one of the varieties $\mathcal{SL}$, $\mathcal{ZM}$ or $\mathcal{SL} \lor \mathcal{ZM}$.

**Proof.** (i) $\Rightarrow$ (ii) immediately follows from Proposition 1.5.

(ii) $\Rightarrow$ (iii). We may assume that $\mathcal{V}$ is a proper variety. Then by Theorem 3.1 $\mathcal{V} = \mathcal{R} \lor (\mathcal{V} \land \mathcal{SL})$ for some Rees variety $\mathcal{R}$. Now suppose that $\mathcal{R} \not\subseteq \mathcal{ZM}$ and choose a semigroup $N \in \mathcal{R}$ which does not satisfy the identity $xy = 0$. Let $\mathcal{W}$ be the variety of all commutative semigroups in $\mathcal{V}$ and $\mathcal{G}$ the semigroup variety generated by a finite non-Abelian group $G$. Then the argument from the proof of Theorem 3.1 shows that

$$\mathcal{W} \lor (\mathcal{G} \land \mathcal{V}) \neq (\mathcal{W} \lor \mathcal{G}) \land \mathcal{V}. \tag{6}$$

Indeed, the left-hand side of (6) is equal to $\mathcal{W}$ and thus consists of commutative semigroups while the right-hand side contains the non-commutative semigroup $(N \times G)/I$ where $I$ is the ideal $\{(0, g) \mid g \in G\}$. The inequality (6) means that $\mathcal{V}$ is not an upper-modular element, a contradiction. Thus, $\mathcal{R} \subseteq \mathcal{ZM}$. Since $\mathcal{ZM}$ is an atom, either $\mathcal{R}$ is trivial or $\mathcal{R} = \mathcal{ZM}$. Hence $\mathcal{V} = \mathcal{R} \lor (\mathcal{V} \land \mathcal{SL})$ either is trivial or coincides with one of the varieties $\mathcal{SL}$, $\mathcal{ZM}$ or $\mathcal{SL} \lor \mathcal{ZM}$.

(iii) $\Rightarrow$ (i) follows from Proposition 2.4 and Lemma 1.6(i). \qed

The classification of the neutral elements of SEM that is a byproduct of Proposition 4.1 and the equivalence (i) $\Leftrightarrow$ (ii) appear to be of independent interest. Observe that, generally speaking, the neutral elements of a lattice
need not coincide with its strongly modular elements (any non-distributive modular lattice may serve as an example).

As a corollary of Proposition 4.1 we may also express the ‘strong non-modularity’ of \( \mathsf{SEM} \) in a more pictorial way. Let us call the dual of the lower heptagon (see Fig. 2) the upper heptagon.

**Corollary 4.2.** Every proper semigroup variety is contained in a sublattice of \( \mathsf{SEM} \) isomorphic to either the pentagon or the lower heptagon or the upper heptagon.

**Proof.** For varieties that are not strongly modular the claim follows from Propositions 1.1 and 1.2 and the dual of the latter proposition. Therefore it suffices to find pentagon sublattices whose bottom varieties are the proper strongly modular elements of \( \mathsf{SEM} \). By Proposition 4.1 these elements are the trivial variety and the varieties \( \mathsf{SL}, \mathsf{ZM} \) and \( \mathsf{SL} \lor \mathsf{ZM} \).

For the trivial variety one of the possible constructions of such a pentagon sublattice is contained in the proof of Proposition 4.1. Namely, let \( \mathcal{R} \) be a Rees variety not contained in \( \mathsf{ZM} \), \( \mathcal{R} \) the variety of all commutative semigroups in \( \mathcal{R} \), and \( \mathcal{G} \) the variety generated by a finite non-Abelian group. Then the meet \( \mathcal{R} \land \mathcal{G} \) is trivial, and from the inequality (6) it readily follows that

\[
\text{Pent}(\mathcal{R}, (\mathcal{R} \lor \mathcal{G}) \land \mathcal{R}, \mathcal{R} \land \mathcal{G}, \mathcal{R} \lor \mathcal{G}, \mathcal{G}).
\]

The top variety \( \mathcal{R} \lor \mathcal{G} \) of the pentagon (7) does contain the variety \( \mathsf{SL} \). As mentioned in the proof of Proposition 2.4, the restriction of the mapping \( \xi_{\mathsf{SL}} : \mathcal{U} \to \mathsf{SL} \lor \mathcal{U} \) to the subvariety lattice \( L(\mathcal{R} \lor \mathcal{G}) \) is a one-to-one homomorphism. The image of the pentagon (7) under this homomorphism yields a pentagon sublattice whose bottom is \( \mathsf{SL} \).

It amounts to a straightforward calculation to verify that the two remaining varieties \( \mathsf{ZM} \) and \( \mathsf{SL} \lor \mathsf{ZM} \) are contained in the following sublattice:

\[
\begin{array}{c}
\text{var}\{xyz = xzy, xy = x^2y = xy^2\} \\
\text{var}\{xyz = xzy = yzx\} \\
\text{var}\{xy = yx\} \\
\mathsf{SL} \lor \mathsf{ZM} \\
\text{var}\{xy = xz\} \\
\mathsf{ZM}
\end{array}
\]

**Figure 3.** The sublattice containing \( \mathsf{ZM} \) and \( \mathsf{SL} \lor \mathsf{ZM} \)
On Fig. 3 one immediately observes two pentagon sublattices whose bottoms are respectively $ZM$ and $SL \lor ZM$. □

By the way, the ‘external’ pentagon on Fig. 3 is exactly one of the two very first examples of non-modular sublattices in $SEM$ mentioned in the introduction—namely, the example from Ježek’s paper [6].

Observe that Corollary 4.2 must restrict to proper varieties. Indeed, if the variety of all semigroups would belong to a pentagon or a heptagon, it would be the top element of the sublattice. This is impossible because the variety is well known to be join-indecomposable [3].

Concluding the subsection, we mention that no analog of Corollary 4.2 holds for diamonds (that is, sublattices isomorphic to the 5-element modular non-distributive lattice). For instance, from Aženštat’s results [1] it follows that $SEM$ contains no diamond sublattice whose bottom element is the trivial variety.

4.2. First order definability in $SEM$. A subset $D$ of a lattice $\langle L, \lor, \land \rangle$ is said to be definable in $L$ if the property of an element of $L$ ‘to belong to $D$’ is expressible in the first order language of lattices. This means that there exists a first order formula $\Phi(x)$ with one free variable such that evaluating the variable at an element $a \in L$ yields a true sentence if and only if $a \in D$.

The deep study of definable subsets of the lattice $SEM$ carried out by Ježek and McKenzie heavily depends on the fact [8, Theorem 1.11] that the set of Rees varieties is definable$^1$. This fact was established in [8] via a sequence of lemmas involving rather complicated and somewhat artificial formulas. We reproduce here only the first of these formulas (used as a building block in further constructions)—just to create a feeling of difficulties that Ježek and McKenzie had to overcome. Following [8], we describe the formula by usual words rather than logical symbols — otherwise it would be hardly readable.

$$\Phi_1(x) \iff x \text{ is a proper modular element, and for every proper modular element } y \text{ satisfying } y > x, \text{ there exists an element } z \geq x \text{ such that there is no largest } t \geq x$$

$$\text{satisfying } z \geq (z \lor y) \land t.$$  

In [8, Lemma 1.8] it was shown that if substituting a variety $\mathcal{X} \in SEM$ makes the formula $\Phi_1(x)$ hold true, then $\mathcal{X}$ is either a Rees variety or the join of a Rees variety with the variety $SL$ of all semilattices. It was not claimed, however, that $\Phi_1(x)$ (or any other formula explicitly written down)

$^1$We recall that Ježek and McKenzie worked with the lattice of equational theories of semigroups, that is, the dual of $SEM$. The elements of the lattice of equational theories corresponding to Rees varieties are called ideal theories in [8]. When reproducing results and constructions from [8], we adapt them to the terminology of the present note.
defines this set of varieties. In contrast, our Theorem 3.1 shows that the set is definable in a natural and transparent way.

**Proposition 4.3.** The formula

\[ \Psi_1(x) ⇔ (\mathcal{L}M(x) \& \mathcal{C}M(x) \& (∃y \ y > x)) \]

defines the set consisting of all Rees varieties together with all their joins with the variety \( SL \).

**Proof.** Indeed, the formula holds true in \( SEM \) if and only if its free variable is evaluated at a proper modular and lower-modular element of \( SEM \). Now Theorem 3.1 applies. □

Now we are in a position to define the set of Rees varieties by a fairly simple formula thus straightening the proof of [8, Theorem 1.11]. First we observe that the variety \( ZM \) is first order definable. This can be done in several ways but we have chosen the one that appears to be the most natural in the framework of our ‘modularity approach’. Clearly, the set of atoms of any lattice with 0 is definable; let \( A(t) \) stand for the corresponding first order formula.

**Lemma 4.4.** The formula

\[ \Psi_2(t) ⇔ (A(t) \& (∀x, y, z \ x, y, z \not\leq t \rightarrow M(x, y, z))) \]

defines the variety \( ZM \).

**Proof.** Being evaluated at an atom \( A \in SEM \), the formula says that all varieties which do not contain \( A \) satisfy the modular law. The fact that the variety \( ZM \) enjoys this property follows from the modularity of the lattice of unions of groups proved by Pastijn [15,16] and independently by Petrich and Reilly [17]. The fact that for any other atom \( A \in SEM \) there exists three varieties which do not contain \( A \) and violate the modular law follows from the proof of Corollary 4.2. Indeed, the elements of the pentagon sublattice (7) avoid all atoms of \( SEM \) except \( ZM \) and the group atoms \( var\{xy = yx, x^py = y\} \), where \( p \) is a prime divisor of the order of the finite non-Abelian group \( G \) in the construction. It is clear that varying \( G \) we can obtain a pentagon sublattice whose elements avoid any prescribed group atom. □

Now in order to distinguish Rees varieties amongst the varieties satisfying \( \Psi_1(x) \), it suffices to write in the first order language the requirement that if our variety contains an atom then the atom is \( ZM \). Thus, we have

**Proposition 4.5.** The formula

\[ \Psi_1(x) \& (∀t \ A(t) \& t \leq x \rightarrow \Psi_2(t)) \]

defines the set of all Rees varieties. □
References


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