DEFINABILITY OF THE VARIETY
GENERATED BY A COMMUTATIVE MONOID
IN THE LATTICE OF COMMUTATIVE
SEMIGROUP VARIETIES

B. M. VERNIKOV

Abstract. Let $M$ be a commutative monoid. We construct a first-order
formula that defines the variety generated by $M$ in the lattice of all com-
mutable semigroup varieties.

A subset $A$ of a lattice $\langle L; \lor, \land \rangle$ is called definable in $L$ if there exists a first-
order formula $\Phi(x)$ with one free variable $x$ in the language of lattice operations
$\lor$ and $\land$ which defines $A$ in $L$. This means that, for an element $a \in L$, the
sentence $\Phi(a)$ is true if and only if $a \in A$. If $A$ consists of a single element, we
speak about definability of this element.

We denote the lattice of all commutative semigroup varieties by $\text{Com}$. A
set of commutative semigroup varieties $X$ (or a single commutative semigroup
variety $X$) is said to be definable if it is definable in $\text{Com}$. In this situation
we will say that the corresponding first-order formula defines the set $X$ or the
variety $X$.

Let $M$ be a commutative monoid. In [10, Corollary 4.8], we provide an ex-
licit first-order formula that defines the variety generated by $M$ in the lattice
of all semigroup varieties. The objective of this note is to modify the argu-
ments from [10] in order to present an explicit formula that defines the variety
generated by $M$ in the lattice $\text{Com}$.

We will denote the conjunction by $\&$ rather than $\land$ because the latter symbol
stands for the meet in a lattice. Since the disjunction and the join in a lattice
are denoted usually by the same symbol $\lor$, we use this symbol for the join and
denote the disjunction by $\lor$. Evidently, the relations $\leq, \geq, < \text{ and } >$ in a
lattice $L$ can be expressed in terms of, say, meet operation $\land$ in $L$. So, we will
freely use these four relations in formulas. Let $\Phi(x)$ be a first-order formula.
For the sake of brevity, we put
\[
\min_x \{ \Phi(x) \} \equiv \Phi(x) \& (\forall y) \left( y < x \rightarrow \neg \Phi(y) \right).
\]
Clearly, the formula $\min_x \{ \Phi(x) \}$ defines the set of all minimal elements of the
set defined by the formula $\Phi(x)$.

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\end{flushright}

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Many important sets of semigroup varieties admit a characterization in the language of atoms of the lattice \( \text{Com} \). The set of all atoms of a lattice \( L \) with 0 is defined by the formula

\[
A(x) \equiv (\exists y)\left( (\forall z) (y \leq z) \& \min_x \{x \neq y\} \right).
\]

A description of all atoms of the lattice \( \text{Com} \) directly follows from the well-known description of atoms of the lattice of all semigroup varieties (see [2, 8], for instance). To list these varieties, we need some notation.

By \( \text{var} \, \Sigma \) we denote the semigroup variety given by the identity system \( \Sigma \). A pair of identities \( wx = xw = w \) where the letter \( x \) does not occur in the word \( w \) is usually written as the symbolic identity \( w = 0 \).

Let us fix notation for several semigroup varieties:

\[
\begin{align*}
A_n & = \text{var} \left\{ x^n y = y, xy = yx \right\} \quad \text{— the variety of Abelian groups} \\
& \quad \text{whose exponent divides } n, \\
\mathcal{SL} & = \text{var} \left\{ x^2 = x, xy = yx \right\} \quad \text{— the variety of semilattices}, \\
\mathcal{ZM} & = \text{var} \left\{ xy = 0 \right\} \quad \text{— the variety of null semigroups}.
\end{align*}
\]

**Lemma 1.** The varieties \( A_p \) (where \( p \) is a prime number), \( \mathcal{SL}, \mathcal{ZM} \) and only they are atoms of the lattice \( \text{Com} \). \( \square \)

Put

\[
\text{Neut}(x) \equiv (\forall y, z)\left( (x \lor y) \land (y \lor z) \land (z \lor x) = (x \land y) \lor (y \land z) \lor (z \land x) \right).
\]

An element \( x \) of a lattice \( L \) such that the sentence \( \text{Neut}(x) \) is true is called neutral. We denote by \( T \) the trivial semigroup variety.

**Lemma 2 ([6, Theorem 1.2]).** A commutative semigroup variety \( V \) is a neutral element of the lattice \( \text{Com} \) if and only if either \( V = \text{COM} \) or \( V = M \lor N \) where \( M \) is one of the varieties \( T \) or \( \mathcal{SL} \), while the variety \( N \) satisfies the identity \( x^2 y = 0 \). \( \square \)

For convenience of references, we formulate the following immediate consequence of Lemmas 1 and 2.

**Corollary 3.** An atom of the lattice \( \text{Com} \) is a neutral element of this lattice if and only if it coincides with one of the varieties \( \mathcal{SL} \) or \( \mathcal{ZM} \). \( \square \)

A semigroup variety \( V \) is called chain if the subvariety lattice of \( V \) is a chain. Clearly, each atom of \( \text{Com} \) is a chain variety. The set of all chain varieties is definable by the formula

\[
\text{Ch}(x) \equiv (\forall y, z) \left( (y \leq x \& z \leq x \implies y \leq z \lor z \leq y) \right).
\]

We adopt the usual agreement that an adjective indicating a property shared by all semigroups of a given variety is applied to the variety itself; the expressions like “group variety”, “periodic variety”, “nil-variety” etc. are understood in this sense.

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1 This notation is justified because a semigroup with such identities has a zero element and all values of the word \( w \) in this semigroup are equal to zero.
Put
\[N_k = \text{var} \{x^2 = x_1x_2 \cdots x_k = 0, xy = yx\} \text{ (} k \text{ is a natural number)},\]
\[N_\omega = \text{var} \{x^2 = 0, xy = yx\},\]
\[N_3^c = \text{var} \{xyz = 0, xy = yx\}\]
(in particular \(N_1 = T\) and \(N_2 = ZM\)). The lattice of all Abelian periodic group varieties is evidently isomorphic to the lattice of natural numbers ordered by divisibility. This readily implies that non-trivial chain Abelian group varieties are varieties \(A_{p^k}\) with prime \(p\) and natural \(k\), and only they. Combining this observation with results of [9], we have the following

**Lemma 4.** The varieties \(A_{p^k}\) with prime \(p\) and natural \(k\), \(SL, N_k, N_\omega, N_3^c\) and only they are chain varieties of commutative semigroups.

Fig. 1 shows the relative location of chain varieties in the lattice \(\text{Com}\).

![Figure 1. Chain varieties of commutative semigroups](image)

Combining above observations, it is easy to verify the following

**Proposition 5.** The set of varieties \(\{A_p \mid p \text{ is a prime number}\}\) and the varieties \(SL\) and \(ZM\) are definable.

**Proof.** By Lemma 1, all varieties mentioned in the proposition are atoms of \(\text{Com}\). By Corollary 3, the varieties \(SL\) and \(ZM\) are neutral elements in \(\text{Com}\), while \(A_p\) is not. Fig. 1 shows that the varieties \(ZM\) and \(A_p\) are proper subvarieties of some chain varieties, while \(SL\) is not. Therefore the formulas
\[
SL(x) \Rightarrow A(x) \& \text{Neut}(x) \& (\forall y) (\text{Ch}(y) \& x \leq y \rightarrow x = y),
\]
\[
ZM(x) \Rightarrow A(x) \& \text{Neut}(x) \& (\exists y) (\text{Ch}(y) \& x < y)
\]
define the varieties \(SL\) and \(ZM\) respectively, while the the formula
\[
\text{GrA}(x) \Rightarrow A(x) \& \neg \text{Neut}(x) \& (\exists y) (\text{Ch}(y) \& x < y)
\]
define the set \(\{A_p \mid p \text{ is a prime number}\}\).
Note that in fact each of the group atoms $A_p$ is individually definable (see Proposition 15 below). The definability of the varieties $SL$ and $ZM$ is mentioned in [4, Proposition 3.1] without any explicitly written formulas.

Recall that a semigroup variety is called *combinatorial* if all its groups are trivial.

**Proposition 6.** The sets of all Abelian periodic group varieties, all combinatorial commutative varieties and of all commutative nil-varieties of semigroups are definable.

**Proof.** It is well known that a commutative semigroup variety is an Abelian periodic group variety [a combinatorial variety, a nil-variety] if and only if it does not contain the varieties $SL$ and $ZM$ [respectively, the varieties $A_p$ for all prime $p$, any atoms except $ZM$]. Therefore, the sets of all Abelian periodic group varieties, all combinatorial commutative varieties and of all commutative nil-varieties are definable by the formulas

\[
\begin{align*}
\text{Gr}(x) &\equiv (\forall y) \ (A(y) &\land y \leq x \longrightarrow \text{GrA}(y)); \\
\text{Comb}(x) &\equiv (\forall y) \ (A(y) &\land y \leq x \longrightarrow \neg \text{GrA}(y)); \\
\text{Nil}(x) &\equiv (\forall y) \ (A(y) &\land y \leq x \longrightarrow \text{ZM}(y))
\end{align*}
\]

respectively.

The claim that the set of all Abelian periodic group varieties is definable in $\text{Com}$ is proved in [4] without any explicitly written formula defining this class.

Identities of the form $w = 0$ are called *0-reduced*. We denote by $\text{COM}$ the variety of all commutative semigroups. A commutative semigroup variety is called *0-reduced in $\text{Com}$* if it is given within $\text{COM}$ by 0-reduced identities only.

**Proposition 7.** The set of all 0-reduced in $\text{Com}$ commutative semigroup varieties is definable.

**Proof.** Put

\[
\text{LMod}(x) \equiv (\forall y, z) \ (x \leq y \longrightarrow x \lor (y \land z) = y \land (x \lor z)).
\]

An element $x$ of a lattice $L$ such that the sentence $\text{LMod}(x)$ is true is called *lower-modular*. Lower-modular elements of the lattice $\text{Com}$ are completely determined in [7, Theorem 1.6]. This result immediately implies that a commutative nil-variety is lower-modular in $\text{Com}$ if and only if it is 0-reduced in $\text{Com}$. Therefore the formula

\[
\text{0-red}(x) \equiv \text{Nil}(x) \& \text{LMod}(x)
\]

defines the set of all 0-reduced in $\text{Com}$ varieties.

The following general fact will be used in what follows.

**Lemma 8.** If a countably infinite subset $S$ of a lattice $L$ is definable in $L$ and forms a chain isomorphic to the chain of natural numbers under the order relation in $L$ then every member of this set is definable in $L$. 
Proof. Let \( S = \{ s_n \mid n \in \mathbb{N} \} \), \( s_1 < s_2 < \cdots < s_n < \cdots \) and let \( \Phi(x) \) be the formula defining \( S \) in \( L \). We are going to prove the definability of the element \( s_n \) for each \( n \) by induction on \( n \). The induction base is evident because the element \( s_1 \) is definable by the formula \( \min_x \{ \Phi(x) \} \). Assume now that \( n > 1 \) and the element \( s_{n-1} \) is definable by some formula \( \Psi(x) \). Then the formula
\[
\min_x \{ \Phi(x) \& (\exists y) \left( \Psi(y) \& y < x \right) \}
\]
defines the element \( s_n \). \( \square \)

The following lemma is a part of the semigroup folklore. It is known at least since earlier 1980’s (see [5], for instance). In any case, it immediately follows from Lemma 2 of [11] and the proof of Proposition 1 of the same article.

**Lemma 9.** If \( V \) is a commutative semigroup variety with \( V \neq \text{COM} \) then \( V = K \lor N \) where \( K \) is a variety generated by a monoid, while \( N \) is a nil-variety. \( \square \)

Let \( C_{m,1} \) denote the cyclic monoid \( \langle a \mid a^m = a^{m+1} \rangle \) and let \( C_m \) be the variety generated by \( C_{m,1} \). It is clear that
\[
C_m = \text{var} \{ x^m = x^{m+1}, xy = yx \}.
\]
In particular, \( C_{1,1} \) is the 2-element semilattice and \( C_1 = SL \). For notation convenience we put also \( C_0 = T \). The following lemma can be easily extracted from the results of [3].

**Lemma 10.** If a periodic semigroup variety \( V \) is generated by a commutative monoid then \( V = G \lor C_m \) for some Abelian periodic group variety \( G \) and some \( m \geq 0 \). \( \square \)

Lemmas 9 and 10 immediately imply

**Corollary 11.** If \( V \) is a commutative combinatorial semigroup variety then \( V = C_m \lor N \) for some \( m \geq 0 \) and some nil-variety \( N \). \( \square \)

Let now \( V \) be a commutative semigroup variety with \( V \neq \text{COM} \). Lemmas 9 and 10 imply that \( V = G \lor C_m \lor N \) for some Abelian periodic group variety \( G \), some \( m \geq 0 \) and some commutative nil-variety \( N \). Our aim now is to provide formulas defining the varieties \( G \) and \( C_m \).

It is well known that each periodic semigroup variety \( \mathcal{X} \) contains its greatest nil-subvariety. We denote this subvariety by \( \text{Nil}(\mathcal{X}) \). Put
\[
D_m = \text{Nil}(C_m) = \text{var} \{ x^m = 0, xy = yx \}
\]
for every natural \( m \). In particular, \( D_1 = T \) and \( D_2 = N_\omega \).

**Proposition 12.** For each \( m \geq 0 \), the variety \( C_m \) is definable.

**Proof.** First, we are going to verify that the formula
\[
\text{All-C}_m(x) \iff \text{Comb}(x) \& (\forall y, z) \left( \text{Nil}(y) \& x = y \lor z \implies x = z \right)
\]
defines the set of varieties \( \{ C_m \mid m \geq 0 \} \) in \( \text{Com} \). Let \( V \) be a commutative semigroup variety such that the sentence \( \text{All-C}_m(V) \) is true. Then \( V \) is combinatorial. Now Corollary 11 successfully applies with the conclusion that \( M = C_m \lor N \) for
some $m \geq 0$ and some commutative nil-variety $N$. The fact that the sentence $\text{All-}C_m(V)$ is true shows that $M = C_m$.

Let now $m \geq 0$. We aim to verify that the sentence $\text{All-}C_m(C_m)$ is true. It is evident that the variety $C_m$ is combinatorial. Suppose that $C_m = M \vee N$ where $N$ is a nil-variety. It remains to check that $N \subseteq M$. We may assume without any loss that $N = \text{Nil}(C_m) = D_m$. It is clear that $M$ is a commutative and combinatorial variety. Corollary 11 implies that $M = C_r \vee N'$ for some $r \geq 0$ and some nil-variety $N'$. Then $N' \subseteq \text{Nil}(C_m) = N$, whence

$$C_m = M \vee N = C_r \vee N' \vee N = C_r \vee N.$$

It suffices to prove that $N \subseteq C_r$ because $N \subseteq C_r \vee N' = M$ in this case. The equality $C_m = C_r \vee N$ implies that $C_r \subseteq C_m$, whence $r \leq m$. If $r = m$ then $N \subseteq C_r$, and we are done. Let now $r < m$. Then the variety $C_m = C_r \vee N$ satisfies the identity $x^r y^m = x^{r+1} y^m$. Recall that the variety $C_m$ is generated by a monoid. Substituting 1 for $y$ in this identity, we obtain that $C_m$ satisfies the identity $x^r = x^{r+1}$. Therefore $C_m \subseteq C_r$ contradicting the inequality $r < m$.

Thus we have proved that the set of varieties $\{C_m \mid m \geq 0\}$ is definable by the formula $\text{All-}C_m(x)$. Now Lemma 8 successfully applies with the conclusion that the variety $C_m$ is definable for each $m$. \qed

**Proposition 13.** For every natural number $m$, the variety $D_m$ is definable.

**Proof.** Every commutative semigroup variety either coincides with $\text{COM}$ or is periodic. Thus the formula

$$\text{Per}(x) = (\exists y) \ (x < y)$$

defines the set of all periodic commutative varieties. In particular, if $\mathcal{X}$ is a commutative variety such that the sentence $\text{Per} \mathcal{X}$ is true then the variety $\text{Nil} \mathcal{X}$ there exists. Put

$$\text{Nil-part}(x, y) = \text{Per}(x) \& y \leq x \& \text{Nil}(y) \& (\forall z) \ (z \leq x \& \text{Nil}(z) \rightarrow z \leq y).$$

Clearly, if $\mathcal{X}$ and $\mathcal{Y}$ are commutative semigroup varieties then the sentence $\text{Nil-part} \mathcal{X}, \mathcal{Y}$ is true if and only if $\mathcal{X}$ is periodic and $\mathcal{Y} = \text{Nil} \mathcal{X}$. Let $C_m$ be the formula defining the variety $C_m$. The variety $D_m$ is defined by the formula

$$D_m(x) = (\exists y) \ (C_m(y) \& \text{Nil-part}(y, x))$$

because $D_m = \text{Nil} \mathcal{C}_m$. \qed

If $\mathcal{X}$ is a commutative nil-variety of semigroups then we denote by $\text{ZR} \mathcal{X}$ the least 0-reduced in $\text{Com}$ variety that contains $\mathcal{X}$. Clearly, the variety $\text{ZR} \mathcal{X}$ is given within $\text{COM}$ by all 0-reduced identities that hold in $\mathcal{X}$. If $u$ is a word and $x$ is a letter then $c(u)$ denotes the set of all letters occurring in $u$, while $\ell_x(u)$ stands for the number of occurrences of $x$ in $u$.

**Lemma 14.** Let $m$ and $n$ be natural numbers with $m > 2$ and $n > 1$. The following are equivalent:

(i) $\text{Nil} \mathcal{A}_n \mathcal{V} \mathcal{X} = \text{ZR} \mathcal{X}$ for any variety $\mathcal{X} \subseteq D_m$;

(ii) $n \geq m - 1$. 

Proof. (i)→(ii) Suppose that \( n < m - 1 \). Let \( \mathcal{X} \) be the subvariety of \( \mathcal{D}_m \) given within \( \mathcal{D}_m \) by the identity

\[
x^{n+1}y = xy^{n+1}.
\]

Since \( n+1 < m \), the variety \( \mathcal{X} \) is not 0-reduced in \textbf{Com}. Note that \( \mathcal{X} \subseteq \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \) because \( \mathcal{X} \) is a nil-variety. The identity (1) holds in the variety \( \mathcal{A}_n \lor \mathcal{X} \), and therefore in the variety \( \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \). But the latter variety does not satisfy the identity \( x^{n+1}y = 0 \) because this identity fails in \( \mathcal{X} \). We see that the variety \( \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \) is not 0-reduced in \textbf{Com}. Since the variety \( \text{ZR}(\mathcal{X}) \) is 0-reduced in \textbf{Com}, we are done.

(ii)→(i) Let \( n \geq m - 1 \) and \( \mathcal{X} \subseteq \mathcal{D}_m \). One can verify that \( \mathcal{A}_n \lor \mathcal{X} = \mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \). Note that this equality immediately follows from [6, Lemma 2.5] whenever \( n \geq m \). We reproduce here the corresponding arguments for the sake of completeness. It suffices to check that \( \mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \subseteq \mathcal{A}_n \lor \mathcal{X} \) because the opposite inclusion is evident. Suppose that the variety \( \mathcal{A}_n \lor \mathcal{X} \) satisfies an identity \( u = v \). We need to prove that this identity holds in \( \mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \). Since \( u = v \) holds in \( \mathcal{A}_n \), we have \( \ell_x(u) \equiv \ell_x(v) \) (mod \( n \)) for any letter \( x \). If \( \ell_x(u) = \ell_x(v) \) for all letters \( x \) then \( u = v \) holds in \( \mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \) because this variety is commutative. Therefore we may assume that \( \ell_x(u) \neq \ell_x(v) \) for some letter \( x \). Then either \( \ell_x(u) \geq n \) or \( \ell_x(v) \geq n \). We may assume without any loss that \( \ell_x(u) \geq n \).

Suppose that \( n \geq m \). Then the identity \( u = 0 \) holds in the variety \( \mathcal{D}_m \), whence it holds in \( \mathcal{X} \). This implies that \( v = 0 \) holds in \( \mathcal{X} \) too. Therefore the variety \( \text{ZR}(\mathcal{X}) \) satisfies the identities \( u = 0 = v \). Since the identity \( u = v \) holds in \( \mathcal{A}_n \), it holds in \( \mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \), and we are done.

It remains to consider the case \( n = m - 1 \). Let \( x \) be a letter with \( x \in c(u) \lor c(v) \) and \( \ell_x(u) \neq \ell_x(v) \). If either \( \ell_x(u) \geq m \) or \( \ell_x(v) \geq m \), we go to the situation considered in the previous paragraph. Let now \( \ell_x(u), \ell_x(v) < m \). Since \( \ell_x(u) \geq n = m - 1 \), \( \ell_x(u) \equiv \ell_x(v) \) (mod \( n \)) and \( \ell_x(u) \neq \ell_x(v) \), we have \( \ell_x(u) = n = m - 1 \) and \( \ell_x(v) = 0 \). The latter equality means that \( x \notin c(v) \).

Substituting 0 for \( x \) in \( u = v \), we obtain that the variety \( \mathcal{X} \) satisfies the identity \( v = 0 \). We go to the situation considered in the previous paragraph again.

We have proved that \( \mathcal{A}_n \lor \mathcal{X} = \mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \). Therefore \( \text{ZR}(\mathcal{X}) \subseteq \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \). If the variety \( \mathcal{X} \) satisfies an identity \( u = 0 \) then \( u^{n+1} = u \) holds in \( \mathcal{A}_n \lor \mathcal{X} \). This readily implies that \( u = 0 \) in \( \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \). Hence \( \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \subseteq \text{ZR}(\mathcal{X}) \). Thus \( \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) = \text{ZR}(\mathcal{X}) \). 

Now we are well prepared to prove the following

**Proposition 15.** An arbitrary Abelian periodic group variety is definable.

**Proof.** Abelian periodic group varieties are exhausted by the trivial variety and the varieties \( \mathcal{A}_n \) with \( n > 1 \). The trivial variety is obviously definable. For brevity, put

\[
\text{ZR}(x, y) \equiv 0\text{-red}(y) \& x \leq y \& (\forall z) (0\text{-red}(z) \& x \leq z \rightarrow y \leq z).
\]

The sentence \( \text{ZR}(\mathcal{X}, \mathcal{Y}) \) is true if and only if \( \mathcal{Y} = \text{ZR}(\mathcal{X}) \). Let \( m \) be a natural number with \( m > 2 \). In view of Lemma 14, the formula

\[
\mathcal{A}_{\geq m-1}(x) \equiv \text{Gr}(x) \& (\forall y, z, t) (\mathcal{D}_n(y) \& z \leq y \& \text{Nil}\text{-part}(x \lor z, t) \rightarrow \text{ZR}(z, t))
\]
defines the set of varieties \( \{ A_n \mid n \geq m - 1 \} \). Therefore the formula

\[ A_n(x) \iff A_{\geq n}(x) \& \neg A_{\geq n+1}(x) \]

defines the variety \( A_n \). \( \square \)

It was proved in [4] that each Abelian group variety is definable in the lattice \( \text{Com} \). However this paper contain no explicit first-order formula defining any given Abelian periodic group variety.

Now we are ready to achieve the goal of this note.

**Theorem 16.** A semigroup variety generated by a commutative monoid is definable.

**Proof.** Let \( V \) be a variety generated by some commutative monoid. According to Lemma 10, \( V = A_n \lor C_m \) for some \( n \geq 1 \) and \( m \geq 0 \). It is easy to check that the parameters \( n \) and \( m \) in this decomposition are defined uniquely. Therefore the formula

\[ (\exists y, z) (A_n(y) \& C_m(z) \& x = y \lor z) \]

defines the variety \( V \) (we assume here that \( A_1 \) is the evident formula defining the variety \( A_1 = T \)). \( \square \)

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**REFERENCES**


**Department of Mathematics and Mechanics, Ural State University, Lenina 51, 620083 Ekaterinburg, Russia**

**E-mail address:** bvernikov@gmail.com